The operad of parenthesized braids and the group GT

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I. Braid groups

II. The operad PaB

III. The gp GT

I.1 Geometric def° of braid gps

Consider $\mathrm{Conf}_n \mathbb{C} \longrightarrow \mathrm{Conf}_n \mathbb{C} / \Sigma_n$

$p_n \in \mathrm{Conf}_n \mathbb{C}$ $\longmapsto$ $\overline{p_n} \in \mathrm{Conf}_n \mathbb{C} / \Sigma_n$

Def $B_n = \pi_1 (\mathrm{Conf}_n \mathbb{C} / \Sigma_n , \overline{p_n})$

$PB_n = \pi_1 (\mathrm{Conf}_n \mathbb{C} , p_n)$

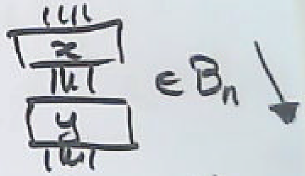
Have a S.E.S.

$1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow \Sigma_n \longrightarrow 1$

View elements in B_n as isotopy classes of n disjoint curves $[0,1] \longrightarrow \mathbb{C} \times [0,1]$

Ex. $b_1 =$  $\in B_2$

$b_1^2 =$  $\in PB_2$

Convention. $xy =$  $\in B_n$

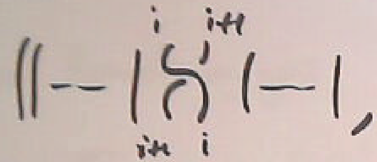
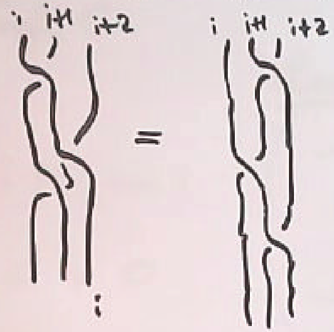
If we view B_n as the groupoid with one object:

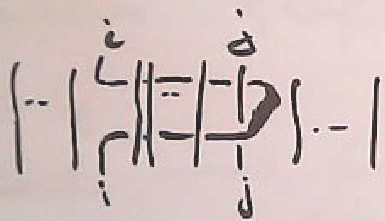
$$xy = y \circ x.$$

* **Algebraic definition**

$$B_n = \left\langle b_i,\ i=1,\dots,n \;\middle|\; \begin{array}{l} b_i b_j = b_j b_i \text{ if } |i-j|>1 \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \end{array} \right\rangle$$

where

$b_i =$  , 

Let $x_{ij} =$ 

for $i<j$.

$$PB_n = \left\langle x_{ij} \;\middle|\; PB1, PB2, PB3 \right\rangle$$

$PB1$: $[x_{ij}, x_{kl}] = 1$ if $\{i,j\}, \{k,l\}$ do not cross

PB2:

$[x_{kj} x_{ij} x_{kj}^{-1}, x_{kl}] = 1$

$i<k<j<l$

PB3:

$[x_{ij}, x_{ik} x_{jk}]$

$= [x_{jk}, x_{ij} x_{ik}]$

$= [x_{ik}, x_{jk} x_{ij}] = 1$

for $i<j<k$.

Relation with free groups:

Fibration: $\mathbb{C} \setminus \{z_1, \dots, z_n\} \longrightarrow \mathrm{Conf}_{n+1}\mathbb{C}$

$$\downarrow$$

$$\mathrm{Conf}_n \mathbb{C}$$

$\Downarrow$

LES on homotopy gps

$$1 \longrightarrow F_n \longrightarrow PB_{n+1} \longrightarrow PB_n \longrightarrow 1$$

since by induction $\pi_2 \mathrm{Conf}_n \mathbb{C} = 0$

It is split by

$$\mathrm{Conf}_n \longrightarrow \mathrm{Conf}_{n+1}$$

$$\vec{z} \longmapsto \left(\vec{z}, 1+\sum |z_i|\right)$$

Hence

$$PB_{n+1} = F_n \rtimes PB_n$$

$$= F_n \rtimes (F_{n-1} \rtimes \cdots (F_2 \rtimes F_1) \cdots).$$

For small n:

$$PB_2 = F(b_1^2) \hookrightarrow F(b_1) = B_2$$


Lemma. Every element in PB_3 is of the form

$$f(b_1^2, b_2^2)\, c^n$$

where $c = (b_1 b_2)^3$, $n \in \mathbb{Z}$, $f \in F_2$.

$c = (b_1 b_2)^3 =$

$\in PB_3$

Sketch of proof:

$$PB_3 = \langle x_{12}, x_{13}, x_{23} \mid R \rangle$$

$$= \langle x_{12}, x_{13}, x_{13}x_{23}x_{12} \mid - \rangle$$

$x_{13} x_{23} x_{12} = c$

$x_{12} = b_1^2$

$x_{13} = b_2 b_1^2 b_2^{-1}$

<u>Fact</u>. c is central in B_3

<u>Fact</u>. $\nexists$ relations between b_1^2 and b_2^2. $\square$

<u>COR</u>. $PB_3 \cong F(b_1^2, b_2^2) \times F(c)$

$PB_3 \longrightarrow PB_2 = F(b_1^2)$

$c \longmapsto b_1^2$

(forget 2^{nd} strand)

$b_1^2 \longmapsto 0$

$b_2^2 \longmapsto 0$

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* <u>Prounipotent completion</u> (in char 0)

$G = \langle S = \{g_1, \dots, g_n\} \mid R \rangle$ fin. pres.

Consider the complete Hopf alg

$H = \mathbb{K}[G]^\wedge_I$, $I = \ker(\varepsilon) = \langle 1 - g_i \rangle$

$$\begin{array}{ccc} & \mathbb{K}[G]^\wedge_I & \\ \subset & & \supset \\ 1 + \hat{I} & \underset{\exp}{\overset{\log}{\rightleftarrows}} \ (\cong) & \hat{I} \\ \cup & & \cup \\ \hat{G}(\mathbb{K}) = \mathrm{Gplike}(H) & \underset{\exp}{\overset{\log}{\rightleftarrows}} \ (\cong) & \hat{\mathfrak{g}} = \mathrm{Prim}(H) \\ \in & & \in \\ g^\lambda = \sum_{k \geq 0} \binom{\lambda}{k} (1-g)^k & & -\log(g) = \sum_{k \geq 1} \frac{1}{k} (1-g)^k \end{array}$$

Descript^n of $\hat{\mathfrak{g}}$: "$\gamma_i = \log g_i$"

$\hat{\mathfrak{g}} = \dfrac{\widehat{\mathrm{Lie}}\{\gamma_1, \dots, \gamma_n\}}{\left(\log R_i(e^{\gamma_1}, \dots, e^{\gamma_n}) = 0\right)}$

$\rightsquigarrow \hat{G}(\mathbb{K}) = \exp(\hat{\mathfrak{g}})$
is the $\mathbb{K}$-pronilpotent completion of G.

Ex

- $\widehat{PB}_3(\mathbb{K}) = \hat{F}_2(\mathbb{K}) \times \mathbb{K}$

$\hat{F}_2(\mathbb{K}) \cong \langle [b_1^2]^{d_1}, [b_2^2]^{d_2} \rangle$, $\mathbb{K} \cong c^{d_3}$

for $d_i \in \mathbb{K}$.

- $\hat{B}_3(\mathbb{K}) \cong \mathbb{K}$ (in char 0)

$\hookrightarrow$ c central, so $b_1 b_2 b_1 = b_2 b_1 b_2 = c^{1/2}$

... thus $b_1 = b_2 = c^{1/6}$. □

Csq. SES

$1 \to \hat{F}(\bar{b}_1, \bar{b}_2) \to \widehat{PB}_3(\mathbb{K}) \to \hat{B}_3(\mathbb{K}) \to 1$

where $\bar{b}_1 = c^{-1/3}\, b_1^2$

$\bar{b}_2 = c^{-1/3}\, b_2^2$.

II. Operad of parenthesized braids

Using the contraction

$$D_2(n) \xrightarrow{\sim} \mathrm{Conf}_n \mathbb{C}$$

we get an operadic composition in $PB = \{PB_n\}_{n \geq 0}$.

Note. $\Sigma_n \curvearrowright PB_n$ is trivial.

$\rightsquigarrow$ PB is a non-Σ operad.

* Cosimplicial structure on PB:

Since $e \in PB_2$ satisfies

$$e \circ_1 e = e \circ_2 e$$

we get a cosimplicial str.

$$d_i : PB_n \longrightarrow PB_{n+1}$$

$$f \longmapsto \quad d_0 f = e \circ_2 f, \quad d_i f = f \circ_i e, \quad d_{n+1} f = e \circ_1 f$$

$s_i : PB_{n+1} \longrightarrow PB_n$ forgets i^{th} strand.

Illustration: $d_i(b_1^2) \in PB_3$

d_0 $\quad$ d_1 $\quad$ d_2 $\quad$ d_3

* Motivation from PaB

If D_2 were an operad in Top_* we would have an operad in Grp.

Pb. $\nexists$ Σ_n-invariant configuration in $D_2(n)$.

What we can do :

consider the groupoid $\pi_{\leq 1} D_2 \in Grpd$.

Pb: Too big.

↳ Solution : PaB "smallest" non-trivial suboperad of $\pi_{\leq 1} D_2$

s.t. $PaB(n) \subseteq \pi_1 D_2(n)$ is full.

$PaB(1) = \{$ ① $\}$

$PaB(2) = \{$ ①②, ②① $\}$

PaB(n) gen. by those.

Introduce $Pa \in Op(Set)$

the operad of parenthesized permutations.

$Pa(n) \subset D_2(n)$

$(1(23))4 \in Pa(4)$

Def $PaB = \pi_1(Conf_n, Pa)$

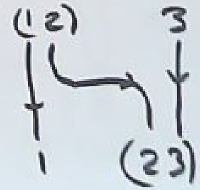
Explicitly: $PaB : FinSet^{\simeq} \longrightarrow Grpd$

is given

$PaB(I) =$
- ob : parenthesized permutations on I
- mor : $Hom_{PaB(I)}(A_I, B_I)$

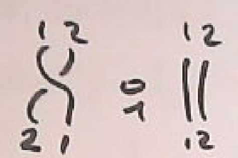
$= \{$ braids preserving the elements $\}$

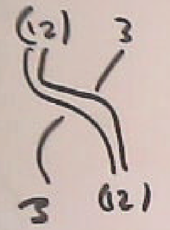
Ex · $\tau =$  $\in Hom_{PaB(2)}(12, 21)$

· $\alpha =$  $\in Hom_{PaB(3)}((12)3, 1(23))$

Composition: $b \circ_i b'$

replace i^{th} strand of b by a thinner version of b'.

Ex. $\tau \circ_1 1\!\!1_{12} =$ 

$=$ 

* We can define $d_i : PaB(n) \to PaB(n+1)$.

(with same formulas, replacing e by $1\!\!1_{12}$).

Rmk. NOT cosimplicial object, because

on objects: $d_1 d_0 (1) = d_1(12) = (12) \circ_1 (12)$

$= (12)3$

$d_0 d_0 (1) = d_0(12) = (12) \circ_2 (12)$

$= 1(23)$.

Note. For any operad $\mathcal{O}$ (in cat.) with the same objects as PaB, we can define

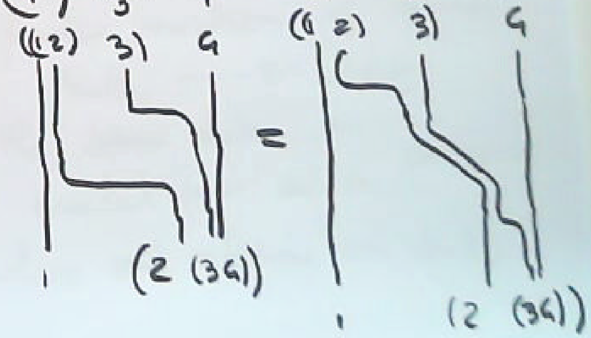
$d_i : \mathcal{O}(n) \longrightarrow \mathcal{O}(n+1)$.

Then any functor morphism $PaB \to \mathcal{O}$ will respect the d_i.

* **Pentagon equation**:

In $PaB(4)$, compute $d_i(\alpha)$ and find

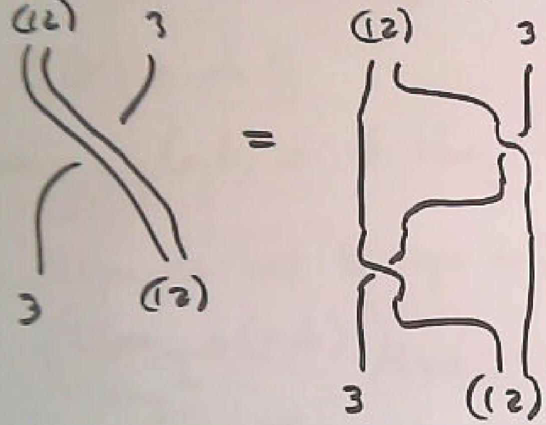
(P) $d_3\alpha \circ d_1\alpha = d_0\alpha \circ d_2\alpha \circ d_4\alpha$.



* Hexagon equations

Relation between $d_i(z)$:

(H1) $d_1 z = (123)\cdot\alpha \circ ((23)\cdot d_3 z) \circ ((23)\cdot\alpha^{-1}) \circ d_0 z \circ \alpha$



$\in \mathrm{Hom}_{PaB(3)}((12)3, 3(12))$

(H2) $d_1 \bar{z} = (123)\cdot\alpha \circ (23)\cdot d_3 \bar{z} \circ (23)\cdot\alpha^{-1} \circ d_0 \bar{z} \circ \alpha$

$\bar{z} = (12)\cdot z^{-1} =$

Thm. For $O \in$ Op Gpd with $ob(O) = ob(PaB)$, then functors $F: PaB \longrightarrow O$ that are id on objects are in bijection with

$(F(z), F(\alpha))$ s.t. (P), (H1), (H2).

↳ Use MacLane coherence thm.

Prop. For $\mathcal{C}$ a category, a ~~functor~~ morphism of operads $PaB_+ \longrightarrow End_{\mathcal{C}}$ is the same data as a braided mon cat on $\mathcal{C}$ (where unit relations are strict).

III $\widehat{GT}(\mathbb{K})$

Malcev completion:

$$\mathrm{Grpd} \xrightarrow{\widehat{(-)}(\mathbb{K})} \mathrm{Cat}(\mathrm{CoAlg}_{\mathbb{K}})$$

$$\mathcal{G} \longmapsto \hat{\mathcal{G}}(\mathbb{K})$$

$$\begin{cases} \mathrm{ob}\, \hat{\mathcal{G}}(\mathbb{K}) = \mathrm{ob}\, \mathcal{G} \\ \mathrm{Hom}_{\hat{\mathcal{G}}(\mathbb{K})}(a,b) = \text{the } \mathrm{Hom}_{\mathcal{G}}(a,b)^{\wedge} \end{cases}$$

Completion is w.r.t. to topo defined by $\left(\mathrm{Hom}_{I^{k}}(a,b)\right)_{k\geq 0}$ where $I^{k} \subseteq \mathcal{G}$ wide subcat. whose mor. lie in the k^{th} power of kernels of counits of the $\mathrm{Hom}_{\mathcal{G}}(a,b)$.

This is mon. sym.

$\rightsquigarrow \widehat{PaB}(\mathbb{K})$.

Def. $\widehat{GT}(\mathbb{K}) := \mathrm{Aut}^{+}_{\mathrm{Op\,Grpd}_{\mathbb{K}}}\left(\widehat{PaB}(\mathbb{K})\right)$

$+$ means that the auto. fix the objects.

Thm (Drinfel'd)

There is a bijection between $\widehat{GT}(\mathbb{K})$ and $(f,\lambda) \in \hat{F}_2(x,y) \times \mathbb{K}^{\times}$ s.t.

$$\begin{cases} \cdot\ f(x,y) = f(y,x)^{-1} \quad \in \hat{F}_2(x,y) \\ \cdot\ f(x_3,x_1)\, x_3^{\mu} f(x_2,x_3)\, x_2^{\mu} f(x_1,x_2)\, x_1^{\mu} = 1 \quad \text{for any } x_1x_2x_3 = 1 \text{ and } \mu := \frac{\lambda-1}{2} \\ \cdot\ f(x_{12}, x_{23}x_{24}) f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34}) f(x_{12}x_{13}, x_{23}x_{24}) f(x_{12}, x_{34}) \quad \in \widehat{PB}_4(\mathbb{K}) \end{cases}$$

The transformed multiplication is

$$(d_1,f_1)\cdot(d_2,f_2) = \left(d_1 d_2,\ f_2\left(f_1(x,y)\,x^{d_1} f_1(x,y)^{-1},\ y^{d_1}\right) f_1(x,y)\right).$$

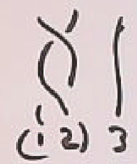
Let $F \in \widehat{GT}(\mathbb{K})$. Then

$\tau^{-1}\circ F(\tau)$ gplike so $= [a^2]^{\mu}$.

$\alpha^{-1}\circ F(\alpha)$ — so $= f(\tilde{b}_1^2, \tilde{b}_2^2)\, c^{?}$.

[For F to be an automor. we need $\mu \neq -\frac{1}{2}$ (so $d \neq 0$). $d := 2\mu+1$]

Here: $b_1 := d_3 \tau^2 =$



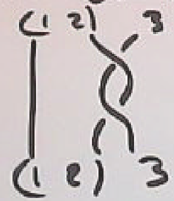
$c := (b_1 b_2)^3$

$\tilde{b}_1^2 := c^{-1/3} b_1^2$

$\tilde{b}_2^2 := c^{-1/3} b_2^2$

$b_2 := \alpha^{-1}\circ d_0(\tau^2)\circ \alpha$

$=$



~~Result~~. Pentagon equation in PaB gives the 3rd Drinfel'd equation. | (H1), (H2) gives the first 2 equations.